A limit-cycle solver for nonautonomous dynamical systems

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Abstract

A numerical technique used to solve boundary value problems is modified to find periodic steady-state solutions of nonautonomous dynamical systems. The technique uses a matrix representation of the time derivative obtained through trigonometric interpolation of a periodic function. Such a differentiation matrix yields exact values for the derivative of a trigonometric polynomial and therefore, can be used as the main ingredient of a numerical method to solve nonlinear dynamical systems. We apply this technique to obtain some limit cycles and bifurcation points of a sinusoidally driven pendulum and the steady-state response of an electric circuit.

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1   Introduction

Dynamical systems can be given by a set of ordinary differential equations and some initial conditions. Such problems are usually solved by conventional procedures such as Runge-Kutta methods, multistep methods or general linear methods (see for example [1]). For instance, shooting methods [2], extrapolation methods [3] and Newton methods based on the Poincare map [4] has been applied to accelerate the convergence of the solution to the periodic steady-state of electric systems. Since some of these implementations requires a considerable computational effort, an alternative and more simple technique may be desired.

In a sequel of papers (see [5] and references therein), a Galerkin-collocation-type method for solving differential boundary value problems has been developed. In some restricted cases this technique yields exact results, i.e., the numerical output can be interpolated to obtain the functions that solve exactly the problem, and therefore it have been used to implement a numerical scheme to solve dynamical systems that consists basically in the substitution of the derivatives appearing in the differential equations by finite-dimensional matrices whose entries depend on certain set of points in a simple form. In some cases, the nodes have to be chosen according to some criterion in order to accelerate the convergence to the solution. The differentiation matrices used in this technique arise naturally in the context of the interpolation of functions.

The aim of this paper is to present a novel method to obtain numerical approximants to the steady-state solutions of nonautonomous systems, based on the use of the differentiation matrix for periodic functions given in [6].

The main idea of the method is established in Sec. 2 and two examples are given in Sec. 3.

2   Discrete formalism

In this section we only present the main results of our discretization scheme for periodic functions; proofs and further applications can be found in [6]. Let D be the $N \times N$ matrix whose components are given by

$$
D_{jk} = \begin{cases} \sum_{l=1}^{N}{}' \dfrac{1}{2} \cot \dfrac{t_j - t_l}{2}, & j = k, \\[2ex] \dfrac{\tau_j}{2\tau_k} \csc \dfrac{t_j - t_k}{2}, & j \neq k, \end{cases}
$$

where $-\pi < t_1 < t_2 < \cdots < t_N \leq \pi$ and

$$
\tau_j = \frac{d}{dt} \Big[\prod_{l=1}^{N} \sin \frac{t - t_l}{2} \Big]_{t_j}.
$$

Let $x(t)$ be a trigonometric polynomial of degree at most n and x and x' denote the $N \times 1$ vectors whose elements are $x(t_j)$ and $x'(t_j)$, i.e., the values of the polynomial and its derivative at the nodes respectively. Then, the matrix D applied to x becomes equal to x' whenever $N = 2n + 1$. Since the derivative of a trigonometric polynomial is again a trigonometric polynomial of the same degree, we have that

$$x^{(k)} = D^k x, \quad k = 0, 1, 2, \ldots, \tag{1}$$

where $x^{(k)}$ is the vector whose elements are given by the kth derivative of $x(t)$ evaluated at the nodes.

The differentiation matrix D takes the simple form

$$D_{jk} = \begin{cases} 0, & j = k, \\ \dfrac{(-1)^{j+k}}{2 \sin \frac{\pi}{N}(j-k)}, & j \neq k, \end{cases} \tag{2}$$

if the nodes are chosen to be the N equidistant points[1]

$$t_j = -\pi + \frac{2\pi j}{N}, \quad j = 1, 2, \cdots, N. \tag{3}$$

If the periodic function $x(t)$ is not a polynomial, a residual vector depending on $x(t)$, k and N must be added to the right-hand side of (1). However, it is expected that the norm of such a vector approaches zero as the number of nodes is increased since $x(t)$ can be expanded in a Fourier series. Therefore, if N is great enough, the residual vector can be ignored and the function $x^{(k)}(t)$ can be approximated by an interpolation of the elements of $D^k x$.

Let us consider now a nonautonomous system of m components described by

$$\dot{x} = f(x, \omega t), \tag{4}$$

where x is the vector of components $x_1(t), x_2(t), \ldots, x_m(t)$ and f is a nonlinear vector function of $m + 1$ variables, periodic in t, with period $T = 2\pi/\omega$. A very important problem is to find the response of the system in the steady-state regime. Taking into account the periodicity of this response, we can use the differentiation matrix (2) to obtain approximants to the steady-state solution of (4) according to the following scheme.

First of all, we need to change $\omega t \to t$ in (4) in order to change the period of f to 2π. Thus, (4) becomes

$$\omega \dot{x} = f(x, t), \tag{5}$$

Let us take an odd number N of points t_j as given by (3) and evaluate (5) at each node to form an equality between $Nm \times 1$ vectors in such a way that the first N entries of the left-hand side are the components of the vector $\dot{x}_1$, i.e., $\dot{x}_1(t_1), \dot{x}_1(t_2), \ldots, \dot{x}_1(t_N)$; the

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[1] The same result is obtained for the set of points $t_j = \pi(2j - N - 1)/N$.

following N entries are the components of $\dot{x}_2$, i.e., $\dot{x}_2(t_1), \dot{x}_2(t_2), \ldots, \dot{x}_2(t_N)$, and so on. Now we can approximate the vectors blocks $\dot{x}_k$ by Dx_k to obtain the discrete form of (5), which can be written as

$$\omega \sum_{l=1}^{N} D_{jl} x_k(t_l) = f_k(x_1(t_j), x_2, (t_j) \ldots, x_m(t_j), t_j), \qquad j = 1, 2, \ldots, N, \tag{6}$$

where $k = 1, 2, \ldots, m$, or in the more compact form

$$\omega \mathbf{D} \mathbf{X} = \mathbf{F}, \tag{7}$$

where $\mathbf{F}$ denotes the $Nm \times 1$ vector whose elements are given by the right-hand side of (6) first running j and then k, $\mathbf{X}$ is the vector whose elements are given by $x_k(t_j)$ (the unknown solution) ordered in a similar way and $\mathbf{D} = 1_m \otimes D$, where 1_m is the identity matrix of dimension m.

The points on which our method is based are the following:

1. If (4) has a limit cycle, then the solution of (7) is an approximation of the steady-state solution of (4).

2. Since (7) [or equivalently (6)] is a system of Nm nonlinear equations with Nm unknowns $x_k(t_j)$, its solution can be obtained by using a standard procedure (Newton's method for instance).

This shows that a nonautonomous system in the steady-state regime can be described approximately by a system of nonlinear algebraic equations. The procedure sketched in these statements is not concerned at all with the initial conditions of the system and yields simultaneously all the values of $x_k(t_j)$ (at all times). Thus, this method is quite different in essence to those designed as initial-value-problem solvers.

To obtain the solution of (6) we can use the Newton method or some variation of it. However, as it is well-known, not always is easy to give a good initial approximation to attain convergence. A good issue is to make a few Runge-Kutta integrations of the system to yield an initial approximation $\mathbf{X}_0$. Since this procedure is time-consuming, a more simple way to find $\mathbf{X}_0$ should be tried, but it will depend on the problem to solve.

Due to the nature of our method to approximate the steady-state solution of (5), an algorithm for this technique will consist necessarily in the algorithm selected to solve the set of nonlinear equations.

In the following section we put this method at work on two important nonautonomous dynamical systems.

3   Test cases

We have chosen a chaotic mechanical system and a electric circuit of actual use as test cases. To find their steady-state solutions, we rewrite the equations describing the dy-

namics of these problems in the discretized form (7) and use a FORTRAN90 program and standard libraries running on a personal computer to solve them.

3.1 Test case 1: the driven pendulum

In spite that the classical pendulum is a very old problem, interest on it is still growing. The pendulum becomes a chaotic system when it is driven at the pivot point. Let us consider a rigid and planar pendulum consisting in a mass attached to a light rod of length l which is vertically driven by a sinusoidal force of the form $-A \cos \omega t$ and damped by a linear viscous force with damping μ as in [7]. If θ denotes the angular displacement of the pendulum measured from the vertically downward position, the equation of motion is

$$\frac{d^2\theta}{dt^2} + a\frac{d\theta}{dt} + (1 + b\cos \omega t)\sin\theta = 0, \tag{8}$$

where

$$a = \frac{2\mu}{\sqrt{lg}}, \qquad b = \frac{A\omega^2}{l}.$$

To compare the steady-state solutions yielded by our procedure with those obtained by other authors we take $0 \leq b \leq 200$ (the driving amplitude) as the control parameter, $a = 0.1$ and $\omega = 17.5$. For values of b near zero, a good initial guess is a vector $\mathbf{X}_0$ whose entries are close to π. To catch more solutions for a given value of b, one can try a vector with a sinusoidal deviation from π. Once we have found a solution $\mathbf{X}(p)$, we proceed to obtain $\mathbf{X}(p + \delta p)$ by using $\mathbf{X}_0 = \mathbf{X}(p)$. We have taken $N = 101$ nodes of the form (3) in this case.

As it is known, this system presents period-doubling and bifurcation. Figure 1 displays these phenomena for solutions of (8) close to the inverted state $\theta = \pi$. To make the drawing more simple, we only plot the maximum and minimum values of the angular displacement $\theta(t)$ as functions of the control parameter b. Other solutions, as the hanging state, are not considered here.

The displayed information is in line with the findings of other authors (see for example [7]), but in our case the problem of stability of the solution is not revised. In Fig. 1, θ_1 is the solution corresponding to the inverted state, θ_2 and θ_3 correspond in essence to the same period-1 solution (one can get the second from the first through a reflection on the axis $\theta = \pi$) and θ_4 and θ_5 are period-2 solutions oscillating both of them about the vertical. The asymmetrical way in which θ_5 oscillates is illustrated in Fig. 2.
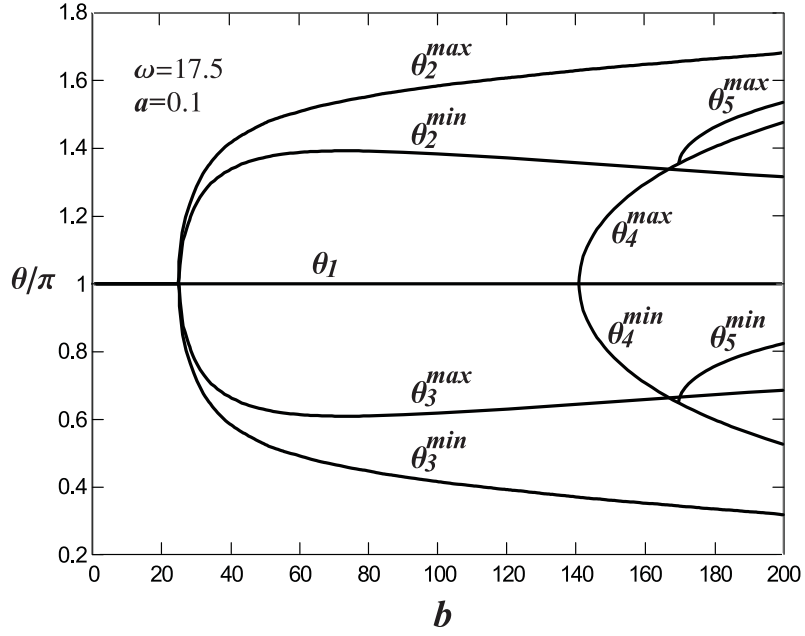
Figure 1: Maximum and minimum values of the angular displacement θ of the vertically driven pendulum (8) vs. the driven amplitude b for $a = 0.1$ and $\omega = 17.5$. For any value of b, the solution $\theta_1 = \pi$ (the inverted pendulum) always was found. The curves labeled by θ_k, $k = 2, \cdots, 5$ correspond to different solutions.
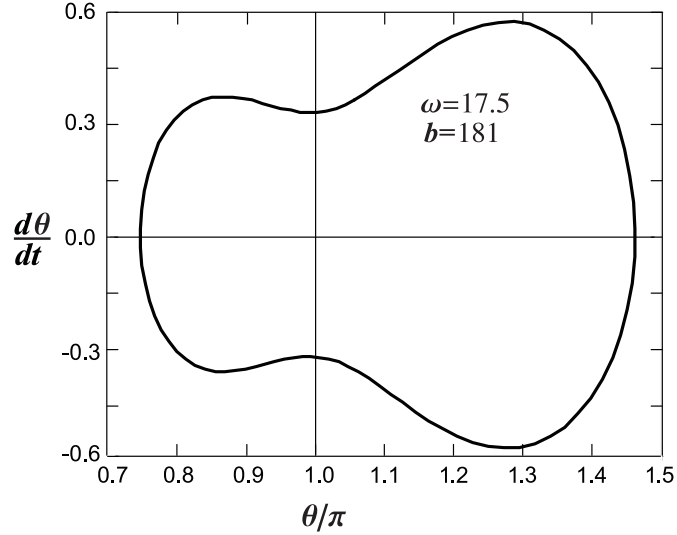

Figure 2: Phase-space diagram of the period-2 "mariachi" solution θ_4 for $a = 0.1$, $\omega = 17.5$ and $b = 181$.

3.2 Test case 2: a commutation circuit

The actual interest in circuits as the one depicted in Fig. 3, relies on the extended use of them in modern power supplies. Usually a primary DC power supply (a battery or the rectified and filtered 60 Hz main line) is commutated at high frequencies (tens of KHz to MHz) in order to obtain a pulse waveform, which can be coupled to the load in some way to obtain a predetermined load voltage. On the circuit shown in Fig. 3, V_s is a simplified model of a transformer secondary, which is followed by a rectifier-filter circuit and a resistive load (R_4). R_1 stands for the series resistance of the diode and R_2 and R_3 model the equivalent series resistance of C_1 and C_2 respectively. R_2 and R_3 are important parameters because they increase ripple voltages and circuit losses.
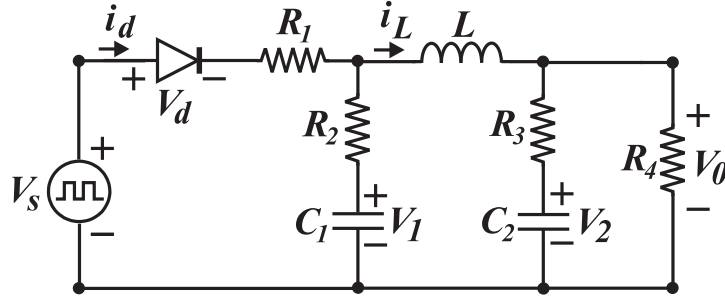

Figure 3: A typical commutation circuit.

The equations for the state variables $V_1 = x_1$, $V_2 = x_2$ and $i_L = x_3$ can be simplified to the form

$$
\begin{aligned}
\dot{x}_1 &= \frac{1}{C_1 R_2}\Big[V_s - x_1 - V_d - i_s R_1\Big(\exp(\frac{V_d q}{\eta k T}) - 1\Big)\Big], \\
\dot{x}_2 &= \frac{1}{C_2(R_3 + R_4)}\Big(-x_2 + R_4 x_3\Big), \\
\dot{x}_3 &= \frac{1}{L}\Big[V_s - \frac{R_4}{(R_3 + R_4)}x_2 - \frac{R_3 R_4}{(R_3 + R_4)}x_3 - V_d - i_s R_1\Big(\exp(\frac{V_d q}{\eta k T}) - 1\Big)\Big].
\end{aligned}
\tag{9}
$$

where

$$
V_d = V_s - C_1(R_1 + R_2)\dot{x}_1 - x_1 - R_1 x_3,
$$

k is the Boltzmann constant, q is the electron charge, T the absolute temperature which is taken at the standard value $T = 300°K$ and η is the emission coefficient. The driving voltage is chosen as a square wave pulse, i.e., $V_s(t) = A_m \text{sgn}(2\pi t/T)$, $t \in [-T/2, T/2]$, and the remaining parameters of this problem are taken at typical values: $A_m = 5.6\text{V}$,

$T = 10^{-5}$s, $i_s = 10^{-8}$A, $R_1 = 0.0149\Omega$, $R_2 = 0.15\Omega$, $R_3 = 0.2\Omega$, $R_4 = 2.0\Omega$, $C_1 = 470.0 \times 10^{-6}$F, $C_2 = 20.0 \times 10^{-6}$F, $L = 20.0 \times 10^{-6}$H, $\eta = 0.8953$.

In order to have results to compare with, we simulated the circuit using a LT version of the Spice program which is currently available for free from Linear Technology, a major semiconductor devices manufacturer. Spice [8] is a standard tool used for electric circuits simulation. Such a program compute the steady-state response in a sequential way and therefore, through a number of periods of the excitation. For this circuit, the response of the system was computed by the Spice simulation over 150 cycles to achieve the steady-state. A maximum step size of 0.04μs was specified. In our case, we use 251 equispaced time values t_j of the form (3) (yielding a constant step size approximately equal to 0.04μs) and all of the steady-state response values were computed at once. Since Spice uses an adaptive strategy for the step size, it is not possible to compare both of the results in a point-by-point basis. However, it comes out that they agree globally up to 10^{-2}.

The output yielded by the present method and the Spice simulation for the more used variables $i_d = x_3 + C_1\dot{x}_1$ and $V_0 = C_2R_3\dot{x}_2 + x_2$, is displayed in Figs. (4a) and (4b), respectively.
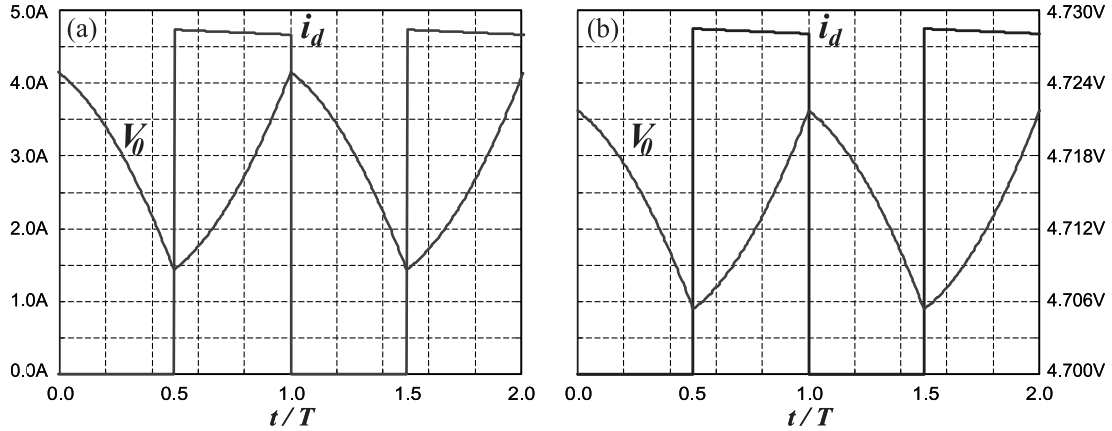

Figure 4: Steady-state solution for the diode current $i_d = x_3 + C_1\dot{x}_1$ and the voltage $V_0 = C_2R_3\dot{x}_2 + x_2$ along two periods obtained by the present method with $N = 251$ nodes (4a) and by a Spice simulation (4b). The left-hand side and right-hand side vertical axes are used for different scales.

4   Final remarks

As the simple tests of the previous section show, the method presented in this paper gives a novel approach to obtain the limit cycles of nonautonomous systems and represents an alternative to the conventional methods of integration. Based on a well-behaved differentiation matrix for periodic functions, it departs from the standard procedures in the way in which the limit cycle is found: the usual integration of the system for long times is

replaced by the problem of finding the solution of a set of nonlinear algebraic equations. This feature makes this method a tool for studying dynamical systems from a new point of view.

5   Acknowledgments

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